OSCILLATIONS OF ELASTIC BODIES WITH FINITE CONDUCTIVITY IN A TRANSVERSE MAGNETIC FIELD

(KOLEBANIIA UPRUGIKH TEL KONECHNOI PROVODIMOSTI V POPERECHNOM MAGNITOM POLE)

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Ia.S. UFLIAND
(Leningrad)

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The investigation of the phenomena of magnetoelasticity, i.e. the oscillations of an elastic body which occur under the action of body forces, of not only mechanical but also electrical origin (the latter arises in case where the motion of the electrical conducting medium takes place in a magnetic field) is the subject of a series of articles by Kaliski [1-6] which consider a perfect conductor, dielectrics, and some other limiting cases as well. A paper by Dolbin [7] is devoted to the propagation of plane waves in a perfect conductor.

Below, some magnetoelastic processes are studied, originating in substances with finite conductivity. With the help of integral transforms, an exact solution is given for the two-dimensional magnetic oscillations problem of an infinite body in a transverse magnetic field under the action of arbitrary body forces.

The system of the original differential equations consists of the dynamic equations of the theory of elasticity containing ponderomotive forces

$$G\Delta\mathbf{u} + (\lambda + G)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{F} + \frac{\mu}{c}\mathbf{j}\times\mathbf{H} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1}$$

and electrodynamic equations for a moving medium (the displacement current is disregarded)

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mu}{c}\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}\right) = \frac{c}{4\pi}\operatorname{rot}\mathbf{h}, \qquad \operatorname{rot}\mathbf{E} = -\frac{\mu}{c}\frac{\partial\mathbf{h}}{\partial t}, \qquad \operatorname{div}\mathbf{h} = 0 \tag{2}$$

In formulas (1) and (2), the following symbols are introduced: $\mathbf{u}$ is the vector displacement, $\mathbf{F}$ is the body force, $\mathbf{H}$ is the applied (homogeneous) magnetic field, $\mathbf{h}$ is the induced magnetic field ($h \ll H$), $\mathbf{E}$ is the electric field, $\mathbf{j}$ is the current density, ρ is the density, λ and G are Lamé coefficients, σ is the conductivity, μ is the permeability and c is the velocity of light.

Below, the two-dimensional problem (where $u_z \equiv 0$, $\partial/\partial z \equiv 0$), is considered, in which it is assumed that the oscillations originate in a transverse magnetic field, so that $\mathbf{H} = H\mathbf{k}$. For these conditions

$$\mathbf{j} \times \mathbf{H} = (c / 4\pi)\,[\mathrm{rot}\,\mathbf{h} \times \mathbf{H}] = -\,(cH / 4\pi)\,\mathrm{grad}\, h_z$$

and the equations of motion assume the form

$$G\Delta\mathbf{u} + (\lambda + G)\,\mathrm{grad\,div}\,\mathbf{u} + \mathbf{F} - \frac{\mu H}{4\pi}\,\mathrm{grad}\, h_z = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{3}$$

If we introduce the scalar potential φ and the vector potential $\mathbf{A} = \psi\mathbf{k}$, by the well-known formula

$$\mathbf{u} = \mathrm{grad}\,\varphi + \mathrm{rot}\,\mathbf{A} \tag{4}$$

and also represent the body force in the form

$$\mathbf{F} = \mathrm{grad}\,\Phi + \mathrm{rot}\,\Psi\mathbf{k} \tag{5}$$

then we obtain separate equations

$$(\lambda + 2G)\triangle\varphi + \Phi - \frac{\mu H}{4\pi}\,h_z = \rho\,\frac{\partial^2\varphi}{\partial t^2}, \qquad G\triangle\psi + \Psi = \rho\,\frac{\partial^2\psi}{\partial t^2} \tag{6}$$

from which it is obvious that the electro-magnetic effects in the given case are observed only for dilatation waves and are not affected by shear waves.

In order to eliminate the magnetic field h_z, the divergence of equation (3) is taken, and we obtain the equation

$$G\,\mathrm{div}\,\triangle\mathbf{u} + (\lambda + G)\triangle\,\mathrm{div}\,\mathbf{u} + \mathrm{div}\,\mathbf{F} - \frac{\mu H}{4\pi}\,\triangle h_z = \rho\,\mathrm{div}\,\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{7}$$

Further, taking the curl of the first equation of (2), and taking into account the second equation, we find

$$\Delta\mathbf{h} = \frac{4\pi\sigma\mu}{c^2}\,\frac{\partial}{\partial t}\,(\mathbf{h} + \mathbf{H}\,\mathrm{div}\,\mathbf{u}) \tag{8}$$

Determining the magnitude Δh_z from this equation, substituting into (7), integrating over time and assuming the initial conditions to be zero, we arrive at the following expression for the projection* of the field h_z:

$$h_z = \triangle \left\{\frac{c^2}{\sigma H \mu^2}\left[(\lambda + 2G)\triangle \int_0^t \varphi\, dt + \int_0^t \Phi\, dt - \rho \frac{\partial \varphi}{\partial t}\right] - H\varphi\right\} \tag{9}$$

Finally, the substitution of this expression into the first equation of (6) allows us to obtain the following equation for the potential φ:

$$(\lambda + 2G + \varkappa)\, \triangle\varphi + \nu\triangle\left[\rho \frac{\partial \varphi}{\partial t} - (\lambda + 2G)\triangle \int_0^t \varphi\, dt\right] + \Phi - \nu\triangle \int_0^t \Phi\, dt = \rho \frac{\partial^2 \varphi}{\partial t^2}$$

$$\left(\varkappa = \frac{\mu H^2}{4\pi},\quad \nu = \frac{c^2}{4\pi\sigma\mu}\right) \tag{10}$$

To obtain the solution of the latter equation, which approaches zero at infinity, together with the first three derivatives, we apply successively to equation (10) the Laplace transform and the second Fourier integral transform, setting

$$f^\circ(\alpha, \beta, p) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y, t)\, e^{-pt + i(\alpha x + \beta y)} dt\, dx\, dy \tag{11}$$

Taking into account the zero initial conditions, as well as the requirements at infinity, we obtain the equation for the transformation of the quantity φ°

$$-(\lambda + 2G + \varkappa)(\alpha^2 + \beta^2)\varphi^\circ - \nu\left[\rho p\, (\alpha^2 + \beta^2)\varphi^\circ + \frac{\lambda + 2G}{p}(\alpha^2 + \beta^2)^2\varphi^\circ\right] +$$

$$+ \Phi^\circ + \frac{\nu}{p}(\alpha^2 + \beta^2)\,\Phi^\circ = \rho p^2\, \varphi^\circ$$

whence

$$\varphi^\circ = \frac{[p + \nu(\alpha^2 + \beta^2)]\,\Phi^\circ}{\rho p^3 + p(\alpha^2 + \beta^2)(\lambda + 2G + \varkappa) + \nu(\alpha^2 + \beta^2)\,[\rho p^2 + (\lambda + 2G)(\alpha^2 + \beta^2)]} \tag{12}$$

Thus, the general solution of the posed problem for the potential φ is given by the inversion formula

$$\varphi(x, y, t) = \frac{1}{8\pi^3 i}\int_L e^{pt} dp \int_{-\infty}^\infty \int_{-\infty}^\infty \varphi^\circ(\alpha, \beta, p)\, e^{-i(\alpha x + \beta y)} d\alpha\, d\beta \tag{13}$$

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* The projections of (8) on the x- and y-axes lead to homogeneous equations for the magnitudes h_x and h_y. From the requirement of the equation that they be zero at infinity, it follows that the quantities $h_x = h_y \equiv 0$.

where L is the infinite line drawn in the complex plane p parallel to the imaginary axis farther to the right than any of the singular points of the function φ°.

For further transforms of the obtained solution, we will assume that the body force is an impulse of magnitude Q, concentrated at an arbitrary point of the plane xOy, such that, without loss of generality, it may be assumed that it is situated at the origin of coordinates and directed along the y-axis. Note that the solution for the case of arbitrary body forces may be obtained from the given path of integration over the region of application of the loads and over time with a subsequent application of Duhamel's integral. Since the potentials Φ and Ψ, corresponding to the body force with the components $F_x = 0$, $F_y = F_y(x, y, t)$, are given by the formulas (see, for example, [8, p.194])

$$\Phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_y(\xi, \eta, t)\,\frac{\partial \ln \rho'}{\partial y}\,d\xi\,d\eta, \quad \Psi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_y(\xi, \eta, t)\,\frac{\partial \ln \rho'}{\partial x}\,d\xi\,d\eta \qquad (14)$$

$$\rho' = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

it follows that for the case of the force $\mathbf{F} = F(t)\mathbf{j}$, concentrated at the origin of coordinates, we have

$$\Phi = \frac{yF(t)}{2\pi(x^2 + y^2)}, \qquad \Psi = -\frac{xF(t)}{2\pi(x^2+y^2)} \qquad (15)$$

Applying to these expressions the Laplace and Fourier transforms, we find for the case of the impulse Q the following transformed expressions:

$$\Phi^{\circ} = \frac{Qi\beta}{\alpha^2+\beta^2}, \qquad \Psi^{\circ} = -\frac{Qi\alpha}{\alpha^2+\beta^2} \qquad (16)$$

Introducing the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $\alpha = R\cos\vartheta$, $\beta = R\sin\vartheta$; the solution of (13) may be represented in the form

$$\varphi = -\frac{Q}{8\pi^3 i}\int_L e^{pt}\,dp\,\frac{\partial}{\partial y}\int_0^{\infty}\int_0^{2\pi} e^{-iRr\cos(\theta-\vartheta)}\,\frac{p+\nu R^2}{R\chi(R,\,p)}\,dR\,d\vartheta \qquad (17)$$

where

$$\chi(R, p) = \rho p^3 + pR^2(\lambda + 2G + \varkappa) + \nu R^2\,[\rho p^2 + (\lambda + 2G)R^2] \qquad (18)$$

With the aid of the well-known integral form of the Bessel function, and integrating over the variable ϑ, we obtain

$$\varphi = \frac{Q}{4\pi^2 i}\int_L e^{pt}dp \int_0^\infty \frac{p + \nu R^2}{\chi(R, p)} J_1(Rr)\, dR \tag{19}$$

Since the denominator χ is a polynomial of the fourth degree relative to the variable R, resolving it into its factors and applying the formula (see [9, p.692])

$$\int_0^\infty \frac{x^2 J_1(ax)\, dx}{x^2 + b^2} = bK_1(ab)$$

where $K_1(z)$ is a Macdonald function, we obtain the solution of the problem in the form of the following complex integral for the potential φ of the dilatation waves

$$\varphi = \frac{Qyt}{2\pi\rho r^2} + \frac{Qy}{4\pi^2 i\rho r}\int_L [R_1(p - \nu R_2^2) K_1(R_2 r) - R_2(p - \nu R_1^2) K_1(R_1 r)] \frac{R_1 R_2}{R_2^2 - R_1^2} e^{pt}\frac{dp}{p^3} \tag{20}$$

$$R_{1,2}^2 = p\,\frac{\gamma \mp \sqrt{\gamma^2 - 4\nu\rho p(\lambda + 2G)}}{2\nu(\lambda + 2G)}, \qquad \gamma = \lambda + 2G + \varkappa + \nu\rho p \tag{21}$$

After determining φ, the value of the induced magnetic field $h_z \equiv h$ is easily obtained on the basis of the first formula in (6)

$$h = \frac{4\pi}{\mu H}\left[(\lambda + 2G)\,\Delta\varphi - \rho\frac{\partial^2\varphi}{\partial t^2} + \Phi\right] \tag{22}$$

Consideration of the asymptotic expressions of the quantities

$$R_2 \approx \frac{p}{a}, \ \left(a = \sqrt{\frac{\lambda + 2G}{\rho}}\right), \quad R_1 \approx \sqrt{\frac{p}{\nu}}, \quad |p| \to \infty$$

shows that for $\nu \neq 0$, the potential φ (as well as the field h) consists of two relatively different components: one of these is a wave propagating with a velocity a, while at the same time the other, in general, does not possess the characteristics of a wave.

The case where $\nu = 0$ corresponding to a perfect conductor, turns out to be singular, since the original equation (10) changes its type. For this

$$R_1 \to \infty, \qquad R_2 \to \frac{p}{a^*}, \qquad a^* = \sqrt{\frac{\lambda + 2G + \varkappa}{\rho}}$$

and the transformation of equation (20) gives

$$\varphi = \frac{Qyt}{2\pi\rho r^2} - \frac{Qy}{4\pi^2 i\rho r a^*}\int_L K_1\left(p\frac{r}{a^*}\right) e^{pt}\frac{dp}{p} \tag{23}$$

If formula [10, Vol.1, p.140] is applied

$$\frac{1}{2\pi i}\int_L K_1\left(p\frac{r}{c}\right)e^{pt}\frac{dp}{p}=\begin{cases}0 & (t<r/c)\\ \sqrt{c^2t^2/r^2-1} & (t>r/c)\end{cases} \tag{24}$$

then, finally, we find

$$\varphi=\frac{Qy}{2\pi\rho r^2}\left(t-\sqrt{t^2-\frac{r^2}{a^{*2}}}\right),\qquad t>\frac{r}{a^*},\qquad \varphi\equiv\frac{Qyt}{2\pi\rho r^2},\quad t<\frac{r}{a^*} \tag{25}$$

The last expression gives the very same law for the potential φ as the ordinary dynamic theory of elasticity (see, for example, [11, Chapt. 12]), where the velocity of propagation has the value [7]

$$\sqrt{\rho^{-1}(\lambda+2G+\mu H^2/\pi)}$$

In this limiting case the magnetic field h is related to the potential φ by the relationship

$$h=-H\Delta\varphi \tag{26}$$

From the general formula (20) we may obtain the solution of the problem for a homogeneous case, when impulses with a density q are equally distributed along the axis Ox. Carrying out in equation (20) the integration over the variable x from $-\infty$ to $+\infty$ and applying formula [9, p. 7]

$$\int_0^\infty K_1\left(a\sqrt{x^2+y^2}\right)\frac{dx}{\sqrt{x^2+|y|}}=\frac{\pi}{2ay}e^{-a|y|},\quad y>0 \tag{27}$$

we obtain

$$\varphi=\frac{qt}{2\rho}+\frac{q}{4\pi\rho i}\int_L\left[R_1^2(p-\nu R_2^2)e^{-R_2|y|}-R_2^2(p-\nu R_1^2)e^{-R_1|y|}\right]\frac{e^{pt}dp}{(R_2^2-R_1^2)p^3} \tag{28}$$

The general expressions (20) or (28) may be reduced to the real form if we consider the singular point of the integrands to be a branch point $p=0$, and carry out integration along the edge of the cut parallel to the negative part of the real axis p.

For obtaining the asymptotic representation of the unknown function for $t\to 0$ and $t\to\infty$, we may apply the well-known method of operational calculus. Thus formula (28) gives the asymptotic expression

$$\varphi\underset{t\to\infty}{\approx}\frac{Q\nu\varkappa}{2(\lambda+2G+\varkappa)^2}\left\{1-\Phi\left[\frac{|y|\sqrt{\lambda+2G+\varkappa}}{2\sqrt{(\lambda+2G)\nu t}}\right]\right\} \tag{29}$$

where $\varphi(z)$ is the probability function.

Simple approximate formulas for the functions φ and h may also be obtained by expanding with respect to small parameters κ, ν and ν^{-1}.

Passing to the limit $\kappa \to 0$, corresponding to the absence of a magnetic field, give $R_1 = p/a$, $R_2 = \sqrt{(p/\nu)}$, which, after using formula (24) leads to the expression (25) with the substitution of velocity a^* by

$$a = \sqrt{\rho^{-1}(\lambda + 2G)}$$

It may be shown that the identical result may also be obtained after passing to the limit $\nu \to 0$, when the elastic body is substituted by a nonconductor.

In conclusion, we give the general expression for the potential of shear waves ψ which, based on equations (6) and (16) are expressed by the formulas

$$\psi = -\frac{Qx}{2\pi\rho r^2}\left(t - \sqrt{t^2 - \frac{r^2}{b^2}}\right), \quad t > \frac{r}{b}, \quad \psi \equiv -\frac{Qxt}{2\pi\rho r^2}, \quad t < \frac{r}{b}, \qquad b = \sqrt{\frac{G}{\rho}} \tag{30}$$

Integrating over x from $-\infty$ to $+\infty$ we may obtain from this the corresponding expression for the one-dimensional case.

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